Sheet 11 Solution

1. We have

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = M \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Multiplying both sides by M^{-1} from the left we have

$$M^{-1} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Substituting this in the given representation of the vector b

$$b = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} (M^{-1}) \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

This gives

$$b = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

or

$$b = \begin{bmatrix} -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Or

$$b = -u_1 - u_2 + 3u_3$$

This problems shows that if the vector components are manipulated as a row vector and we have the transformation:

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = M \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Then a vector

$$a = \begin{bmatrix} \propto_1 & \propto_2 & \propto_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Is transformed to

$$b = \left(\begin{bmatrix} \propto_1 & \propto_2 & \propto_3 \end{bmatrix} (M^{-1}) \right) \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

And

$$a = \left(\begin{bmatrix} \beta_1 & \beta_2 & \beta_3 \end{bmatrix} (\mathrm{M}) \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} \propto_1 & \propto_2 & \propto_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

In this sense, the transformation matrix from a representation in v to a representation in u is $(\mathrm{M})^{-1}$ and the transformation matrix from a representation in u to a representation in v is(M). The order of multiplication is as shown above

2. We have

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ Q_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix} = M \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix}$$

$$b = [1 \;\; 2 \;\; 3 \;\; 0](M^{-1}) \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ Q_0 \end{bmatrix}$$

This gives

$$b = [1 \;\; 2 \;\; 3 \;\; 0] \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ Q_0 \end{bmatrix}$$

Or

$$b = [-1 \;\; -1 \;\; 3 \;\; 0] \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ Q_0 \end{bmatrix}$$

Or

$$b = -u_1 - u_2 + 3u_3$$

3. We have

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ Q_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix} = M \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix}$$

$$b = [1 \;\; 2 \;\; 3 \;\; 0](M^{-1}) \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ Q_0 \end{bmatrix}$$

This gives

$$b = [1 \;\; 2 \;\; 3 \;\; 0] \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 1 & -3 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ Q_0 \end{bmatrix}$$

Or

$$b = [-1 \;\; -1 \;\; 3 \;\; 0] \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ Q_0 \end{bmatrix}$$

Or

$$b = -u_1 - u_2 + 3u_3$$

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4. We have

$$\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}=\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=M\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}$$

Multiplying both sides by M^{-1} from the left we have

$$M^{-1}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}=\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}$$

Substituting this in the given representation of the vector b

$$b=\begin{bmatrix}1\\2\\3\end{bmatrix}^T(M^{-1})\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}$$

This gives

$$b=\left(\left(\begin{bmatrix}1\\2\\3\end{bmatrix}^T(M^{-1})\right)^T\right)^T\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}=\left((M^{-1})^T\begin{bmatrix}1\\2\\3\end{bmatrix}\right)^T\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}=\left((M^T)^{-1}\begin{bmatrix}1\\2\\3\end{bmatrix}\right)^T\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}$$

or

$$b=\left(\begin{bmatrix}1&-1&0\\0&1&-1\\0&0&1\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}\right)^T\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}$$

Or

$$b=-u_1-u_2+3u_3$$

Note: This problem is the same as problem 1 but it emphasis the order of multiplication. It shows that if the vector components are manipulated as a column vector and we have the transformation:

$$\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}=M\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}$$

Then a vector

$$a=\begin{bmatrix}\propto_1\\\propto_2\\\propto_3\end{bmatrix}^T\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}$$

Is transformed to

$$b=\left((M^T)^{-1}\begin{bmatrix}\propto_1\\\propto_2\\\propto_3\end{bmatrix}\right)^T\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}=\begin{bmatrix}\beta_1\\\beta_2\\\beta_3\end{bmatrix}^T\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}$$

And

$$a=\left((M^T)\begin{bmatrix}\beta_1\\\beta_2\\\beta_3\end{bmatrix}\right)^T\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}\propto_1\\\propto_2\\\propto_3\end{bmatrix}^T\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}$$

In this sense, the transformation matrix from a representation in v to a representation in u is $(M^T)^{-1}$ and the transformation matrix from a representation in u to a representation in v is (M^T). The order of multiplication is as shown above

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5. We have

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ Q_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix} = M \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix}$$

And

$$a = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}^T \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix}$$

Then

$$b = \left((M^T)^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} \right)^T \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ Q_0 \end{bmatrix} = \left(\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} \right)^T \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ Q_0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 3 \\ 0 \end{bmatrix}^T \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ Q_0 \end{bmatrix}$$

6. We have

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ Q_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix} = M \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix}$$

And

$$a = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}^T \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix}$$

Then

$$b = \left((M^T)^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} \right)^T \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ Q_0 \end{bmatrix} = \left(\begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} \right)^T \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ Q_0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 3 \\ 0 \end{bmatrix}^T \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ Q_0 \end{bmatrix}$$

7.
- The transformation matrix are as transpose of each other in the two cases
- The new row data is calculated by multiplying the old row data by the transformation matrix or its inverse in a left to right order
- The new vector data is calculated by multiplying the transformation matrix or its inverse by the old vector data in a left to right order
- While working in OpenGL, remember that the internal representation is so that the vector data layout is used.

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = M \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \text{ or } \boldsymbol{u} = M\boldsymbol{v}$$

$$\boldsymbol{a} = [\propto_1 \quad \propto_2 \quad \propto_3] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \boldsymbol{\alpha}^T \boldsymbol{v} \quad \textbf{where} \quad \boldsymbol{\alpha} = \begin{bmatrix} \propto_1 \\ \propto_2 \\ \propto_3 \end{bmatrix}$$

$$\boldsymbol{b} = [\beta_1 \quad \beta_2 \quad \beta_3]\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \boldsymbol{\beta}^T\boldsymbol{u} \ \ \textbf{where} \ \ \boldsymbol{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} \, is\,the\,representation\,of\,\boldsymbol{a}\,w.r.t\,\boldsymbol{u}$$

$$\boldsymbol{\beta}^T = \left(\boldsymbol{\alpha}^T(M^{-1})\right)\, and \ = \boldsymbol{\alpha}^T = \boldsymbol{\beta}^T(M)$$

$$\beta = \left(((M^T)^{-1})\alpha\right)\, and \ = \alpha = (M^T)\beta$$

Note the row/vector data layout, transformation matrix and the order of matrices in multiplication. OpenGl thinks always in terms vector data layout. This is also true for homogenous 4-D coordinates